An energy-conserved solitonic cellular automaton

1992 J. Phys. A: Math. Gen. 25 3369

(http://iopscience.iop.org/0305-4470/25/11/037)

J. Phys. A: Math. Gen. 25 (1992) 3369–3381. Printed in the UK

An energy-conserved solitonic cellular automaton

Zhuhan Jiang†

Department of Mathematics, University of Manchester Institute of Science and Technology, PO Box 88, Manchester M60 1QD, UK

Received 28 November 1991

Abstract. An energy-conserved soliton cellular automaton is proposed. It is a generalization of PSTs model and is shown to contain richer solitonic phenomena. A systematic comparison of their collision statistics is also given.

1. Introduction

Since the observation of the soliton-like behaviour in a new kind of cellular automaton (CA) [1] mainly through computer simulation, substantial research work has been carried out in this connection. This kind of CA, or rather filter CA, typically consists of a collection of *bit* states a_i^t, $-\infty < i < \infty$, $0 \leqslant t \leqslant \infty$, $i, t \in \mathbb{Z}$, the set of integers, by a rule $\mathfrak{F}$ of the form

$$a_i^{t+1} = \mathfrak{F}(a_{i-r}^{t+1}, a_{i-r+1}^{t+1}, \ldots, a_{i-1}^{t+1}, a_i^t, \ldots, a_{i+r}^t) \tag{1}$$

with radius $r > 0$ and $\mathfrak{F}(0, \ldots, 0) = 0$. CA or filter CA recently emerge as non-numerical models for nonlinear phenomena [2, 3]. PST [1] introduced a particular CA

$$a^t = \ldots 0 \ldots 0\, a_0^t \ldots a_i^t \ldots a_L^t\, 0 \ldots 0 \ldots \qquad L < \infty \qquad a_i^t \in \{0, 1\} \tag{2}$$

where $a_0^t = a_L^t = 1$. Its evolution is determined by

$$a_i^{t+1} = \begin{cases} 1 & S_i \neq 0 \text{ even} \\ 0 & \text{otherwise} \end{cases} \qquad S_i = \sum_{j=1}^{r} a_{i-j}^{t+1} + \sum_{j=0}^{r} a_{i+j}^t. \tag{3}$$

A fast rule theorem (FRT) was developed in [4] to facilitate the calculation of the evolution. It was one of the fundamental tools employed to establish stability [4] and many other analytic results [5–7]. In particular, all simple particle collisions are solitonic [6] and there may exist energy loss in the particle evolutions [7]. By simple particle, we mean a particle, a local pattern, containing only one non-zero basic string (BS), while a BS is $(r+1)$ contiguous bits $a_i^t \ldots a_{i+r}^t$. Breathers are also found and it is observed [7] that all seeds (initial pattern states) for CA (3) evolve into periodic particles with faster ones on the right. Recently several new CA have been found for multi-dimensions [8], for pure solitons only [9], as well as for a time-reversible generalization [10] of CA (3).

In this work, however, we derive a new filter CA by modifying the difference equation [7] of PSTs model. This new CA has an irreversible evolution yet keeps its energy conserved. In fact we shall establish several theorems regarding fast rules, asymptotics

† Supported by SERC.

and stability, as well as exhaustive collisions for the evolution. Through theories and computer simulations, we find for our new CA: (i) its energy [11] is always conserved; (ii) all evolution are stable and total pattern widths in the evolution are well bounded; (iii) all seeds evolve into a collection of periodic particles with faster (rightward defined as the positive speed or displacement) ones on the right; (iv) simple particle collisions are solitonic (proof in [6] with some modifications); (v) all energy-conserved evolutions, in particular all existing solitonic collisions, of PSTs model map to a counterpart in our new CA; (vi) a much higher proportion of solitonic interactions among multiple-BS particles exist in the new CA than in PSTs original one. To conclude this section, we remark that although the term solitonic is often a paraphrase of many soliton related features [12–14] such as ISTs [14, 15] and complete integrability [13, 14], it is here specifically referred to solitonic interactions.

2. A new filter CA and its FRT

In comparison with the difference equation [7] for PSTs CA (3)

$$a_n^{t+1} = \left\{ a_{n+r}^t + \sum_{j=0}^{r-1} (a_{n+j}^t - a_{n-r+j}^{t+1}) + \max(a_{n-r}^{t+1}, \ldots, a_{n-1}^{t+1}, a_n^t, \ldots, a_{n+r}^t) \right\} \bmod(2) \tag{4}$$

our new filter CA bears a great resemblance:

$$a_n^{t+1} = \left\{ a_{n+r+1}^t + \sum_{j=1}^{r} (a_{n+j}^t - a_{n-r-1+j}^{t+1}) + \max(a_{n-r}^{t+1}, \ldots, a_{n-1}^{t+1}, a_n^t, \ldots, a_{n+r+1}^t) \right\} \bmod(2) \tag{5}$$

where $a_n^{t+1} = 0$ is as in (4) assumed for n sufficiently far to the left. We note that (5) is essentially of form (1) and therefore a filter CA. Although this new CA was first formulated via a complicated set of rules designed to shut down the energy-leaking 'doors', giving a neater form (5), a much simpler and more symmetric rule for (5) can be shown as [16]

$$a_i^{t+1} = \begin{cases} 0 & T_i = 0 \\ a_i^t & T_i \text{ odd} \\ \overline{a^t} & T_i \neq 0 \text{ even} \end{cases} \qquad T_i = T_i(\boldsymbol{a}^t) \equiv \sum_{j=1}^{r} a_{i-j}^{t+1} + \sum_{j=0}^{r+1} a_{i+j}^t \tag{6}$$

where the *conjugate* $\bar{x}$ of any bit state x is defined by $\bar{x} = 0$ if $x = 1$ and $\bar{x} = 1$ if $x = 0$. To formulate an FRT for (5), we call i a *prebox* position of $\boldsymbol{a}^t$ if

$$a_{i-r-1+j}^{t+1} = a_{i+j}^t \qquad j = 1, \ldots, r \tag{7}$$

and call a prebox position a *box* position if $T_i \neq 0$. Then we have the following lemma.

Lemma 1. (*A*) If i is a prebox position of $\boldsymbol{a}^t$ and $T_i(\boldsymbol{a}^t) = 0$, then $i+1$ is also a prebox position.

(*B*) If i is a box position of $\boldsymbol{a}^t$, then $i+r+1$ is a prebox position and $a_i^{t+1} = \overline{a_{i+r+1}^t}$.

Proof. For the proof of case (*A*), we see that the second and third terms of the RHS of (5) for $n = i$ are zero, which gives $a_i^{t+1} = a_{i+r+1}^t$. This combined with (7) implies (7) also holds when i is replaced by $i+1$. To prove case (*B*), we notice that $T_i(a^t) \neq 0$ due to the box position i, we thus have from (5)

$$a_i^{t+1} = \left\{ a_{i+r+1}^t + \sum_{k=1}^{r} (a_{i+k}^t - a_{i-r-1+k}^{t+1}) + 1 \right\} \bmod(2) = \overline{a_{i+r+1}^t}$$

and hence for $j=1,\ldots,r$

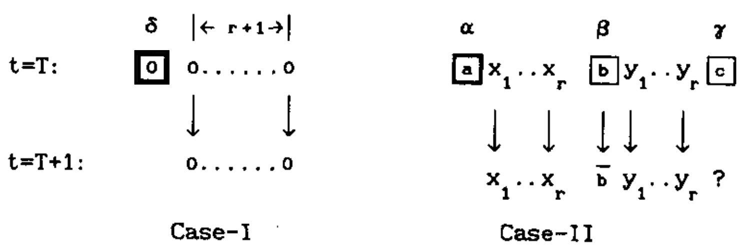
$$a_{i+j}^{t+1}=\left\{a_{i+r+1+j}^{t}+\sum_{k=1}^{r}\left(a_{i+k+j}^{t}-a_{i-r-1+k+j}^{t+1}\right)+1\right\}\operatorname{mod}(2) \tag{8}$$

because the max($\cdot$) in (5) contains both a_i^{t+1} and a_{i+r+1}^{t} which are conjugate to each other. Equation (8) thus implies inductively

$$\begin{aligned} a_{i+j}^{t+1}=\Bigg\{a_{i+r+1+j}^{t}&+\sum_{k=1}^{r-j}\left(a_{i+k+j}^{t}-a_{i-r-1+k+j}^{t+1}\right)+\left(a_{i+r+1}^{t}-a_{i}^{t+1}\right)\\ &+\sum_{k=r-j+2}^{r}\left(a_{i+k+j}^{t}-a_{i-r-1+k+j}^{t+1}\right)+1\Bigg\}\operatorname{mod}(2)=a_{i+r+1+j}^{t} \end{aligned} \tag{9}$$

because the second and fourth (summation) terms are zero by (7) and the induction assumption $a_{i+k}^{t+1}=a_{i+k+r+1}^{t}$ $(1\leqslant k<j)$. Equation (9) thus completes our proof of case (B) and hence lemma 1. □

We now start with a prebox position, say δ (see the diagram below), on the far left and proceed to the right by lemma 1 via finding: prebox → next box if any → next prebox if any. To be precise, we stay on a left-moving frame of speed of $r+1$ and start with a trivial prebox position δ (case I), moving right until we meet a non-trivial prebox position α in case II where $T_\alpha\neq 0$ (particularly so if bit $b\neq 0$ there). Thus α is in fact a box position which implies via lemma 1 that position β ($r+1$ bits away on the right from α) is a prebox position and the bit b there is changed to $\bar{b}$ while the following r bit's y_is are retained intact during the next step of evolution. Since β is now also a prebox position, we can carry on rightwards with this procedure by regarding β as our new α position until a new β is not a box position. In such a case, β becomes a trivial prebox position of δ (case I) type which means we can start the same process rightwards again. By collecting all such β boxes to the following FRT, the evolution of CA (5) can be summarized by theorem 1 below.



Theorem 1 (FRT). The procedure for obtaining the evolution of CA (5) is:

(i) start with any bit position, the left-hand side of which has all bits $a_i^t=0$;

(ii) if all right-hand side bits are zero, go to step (iv). Otherwise move to the right and place a box on the first encountered non-zero bit;

(iii) if the current box and the following r consecutive bits are all zero, go to step (ii). Otherwise put a new box on the $(r+1)$th position on the right of the current box and then go to step (ii);

(iv) conjugate the boxed bits, then move all bits to the left by $r+1$ positions.

For example, a pattern of $*000\boxed{1}101\boxed{0}00000\boxed{1}011\boxed{0}$ for $r=3$ at $t=0$ will evolve via the FRT into $*\boxed{1}011\boxed{0}000000\boxed{1}110\boxed{0}$ (with its new boxes placed there) at $t=1$ where * represents a fixed spatial reference position. We note that (6) can be shown to be equivalent to (5) by proving [16] it holds for lemma 1 and the above FRT. Also *all* non-zero 1-BS particles are non-trivial periodic particles (cf [4-7]) with ℓ (=number of its 1s) a multiple of its period, because in ℓ steps of evolution, the original pattern will re-emerge with a displacement of $(r+1)(\ell-1)$.

3. Evolution asymptotic and energy conservation

Before turning to particle collisions, let us recall that a particle is a finite collection of BSs. Two particles will be called *same* if one particle pattern can evolve into the other up to a displacement. A pattern state a^t is said to have a *splitting* if, after applying FRT ((i)–(iii)), there are $r+1$ consecutive zero bits on the immediate right of a *zero* box position and on the left or *another* box. Obviously this splitting condition implies that defined in [5, 6] for PSTs model, but is stronger. A non-trivial pattern state will always consist of exactly $N+1$ ($N \geqslant 0$) contiguous subpatterns which are split from each other; if M of these N splittings disappear in the next step of evolution, we say that there are M *linking-ups* in the next step. Furthermore we shall, without loss of generality, from now on study the evolution of (5) on a moving frame (towards left) of a constant speed $r+1$ unless otherwise stated. On this frame, the second half of step (iv) in the FRT is effectively removed.

Suppose a pattern state $\boldsymbol{a}=\Sigma_{k=1}^{N}\ \boldsymbol{b}_k=\boldsymbol{b}_1+\ldots+\boldsymbol{b}_N$ (such a summation always denotes that $\boldsymbol{b}_k$ is on the LHS of $\boldsymbol{b}_{k+1}$ for all k and $\boldsymbol{b}_k$s are disjointed contiguous subpatterns of $\boldsymbol{a}$) splits into $\boldsymbol{b}_k$ via the splitting condition such that all $\boldsymbol{b}_k$ are non-splitting particles. We define the raw width $\mathbb{W}(\boldsymbol{a})$ of $\boldsymbol{a}$ as the minimum number of consecutive BSs needed to cover all of its non-zero bits and define the pattern width $W(\boldsymbol{a})=\Sigma_k \mathbb{W}(\boldsymbol{b}_k)$. Hence $W(\boldsymbol{a})\leqslant\mathbb{W}(\boldsymbol{a})$ and the equality holds only if no splitting exists in $\boldsymbol{a}$, in which case $W(\boldsymbol{a})$ is the number of BSs between the first and last boxes placed there by the FRT.

Theorem 2 (stability theorem). (a) All evolved pattern widths of any seed are bounded.

(b) If a particle $\boldsymbol{a}$ creates M linking-ups and N new splittings during the next step, then the width of the pattern increases by at most $M-N$, i.e. $W(\boldsymbol{a}^{t+1})\leqslant W(\boldsymbol{a}^t)+M-N$.

(c) If there are no splittings throughout the evolution of a particle $\boldsymbol{a}$, then it evolves into a periodic particle.

(d) If particle $\boldsymbol{a}^t$ at $t=t_i$ consists of two periodic subpatterns $\boldsymbol{b}$ and $\boldsymbol{c}$ with a distance $d(t_i)$ (the bit count from the last box of particle $\boldsymbol{b}$ on the left to first box of $\boldsymbol{c}$) such that $d(t_i)\to\infty$. Then the speed of particle $\boldsymbol{c}$ is higher than that of $\boldsymbol{b}$.

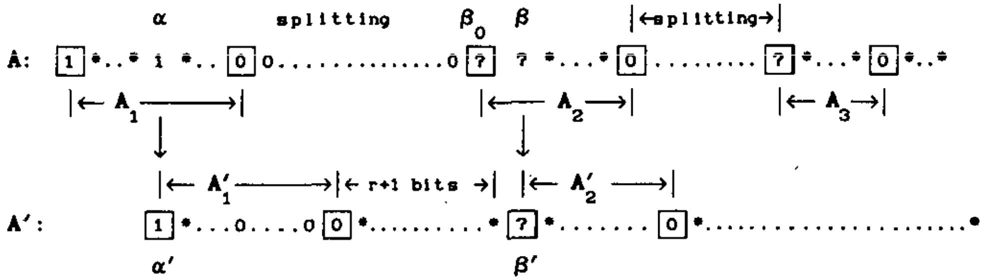
Cases (a) and (b) imply that the evolution is *very* stable in comparison with the stability defined in [4]. Case (c) implies periodic particles are to some extent the natural stable form, while case (d) shows that two periodic particles will have the faster one on the right if their distances during the evolution are unbounded.

Proof. We prove case (c) first. If there are no splittings throughout the evolution of a particle $\boldsymbol{a}$, $2^{W(\boldsymbol{a})\cdot(r+1)}+1$ steps of evolution would more than exhaust all possible pattern configurations if no pattern were repeated. Thus one of these patterns will reappear and therefore $\boldsymbol{a}$ evolves into a periodic particle. In case (d), suppose the speed of $\boldsymbol{b}$ is larger than that of $\boldsymbol{c}$, we let seeds $\boldsymbol{a}^t$ evolve from $t=t_i$, and denote $\boldsymbol{d}_i$ the (evolved) pattern of $\boldsymbol{a}^{t_i}$ just before $\boldsymbol{b}$ and $\boldsymbol{c}$ are to collide in the next step. We choose an infinite subsequence $\{\boldsymbol{e}_j\}$ of $\boldsymbol{d}_i$ such that all $\boldsymbol{e}_j$ are pattern states at different time t_{i_j}. Since $W(\boldsymbol{e}_j)$ are bounded because $\boldsymbol{b}$ and $\boldsymbol{c}$ are periodic, and the sequence is infinite, there will be duplications of the patterns $\boldsymbol{e}_j$. This means $\boldsymbol{a}^t$ evolves into a periodic state, contradicting $d(t_i)\to\infty$. For the same reason the speeds of $\boldsymbol{b}$ and $\boldsymbol{c}$ cannot be the same. We now prove case (a) via case (b). Suppose that the evolution of $\boldsymbol{a}^t$ into $\boldsymbol{a}^{t+1}$ links up M_t split particles but incurs N_t new splittings, then $W(\boldsymbol{a}^{t+1})\leqslant W(\boldsymbol{a}^t)+M_t-N_t$. Since the pattern width $W(\boldsymbol{a}^t)$ changes only if there are new splittings or linking-ups, and furthermore a linking-up together (at the same time or later time) with a splitting cannot increase the pattern width, there can be at most $\mathbb{W}(\boldsymbol{a}^0)$ more linking-ups than splittings. Hence $W(\boldsymbol{a}^t)\leqslant W(\boldsymbol{a}^0)+\Sigma_{k=0}^{t-1}(M_k-N_k)\leqslant W(\boldsymbol{a}^0)+\mathbb{W}(\boldsymbol{a}^0)$ are well bounded.

In the four steps to be followed for the proof of case (b), we use capital bold letter to represent a pattern state at time t and the corresponding lowercase bold letter for the pattern state at time $t+1$.

(i) If particle $\boldsymbol{A}=\Sigma_{i=1}^{N+1}\boldsymbol{A}_i$ has exactly N splittings separating subpatterns $\boldsymbol{A}_i$, if furthermore $\boldsymbol{A}'$ is a non-splitting contiguous subpattern of $\boldsymbol{A}$, then $W(\boldsymbol{A}')\leqslant W(\boldsymbol{A})+N$.

Proof. If $N=0$ then applying the FRT to $\boldsymbol{A}$ directly is sufficient. For $N\geqslant 1$, we apply the FRT to both $\boldsymbol{A}$ and $\boldsymbol{A}'$ (see the diagram below), then $\boldsymbol{A}_1'=\boldsymbol{A}'\cap\boldsymbol{A}_1$ is a subpattern of $\boldsymbol{A}_1$ and hence $W(\boldsymbol{A}_1')\leqslant W(\boldsymbol{A}_1)$. Notice that there is a position β or β_0 (inside $\boldsymbol{A}_2$) which becomes a boxed position β' of $\boldsymbol{A}'$ and the width (from the leftmost box α' of $\boldsymbol{A}'$ through β') $\leqslant W(\boldsymbol{A}_1)+1$. Since the pattern from β' rightwards is again a subpattern of $\boldsymbol{A}_2+\ldots+\boldsymbol{A}_{N+1}$. Inductively we obtain $W(\boldsymbol{A}')\leqslant W(\boldsymbol{A}_{N+1}')+\Sigma_{i=1}^{N}(W(\boldsymbol{A}_i')+1)\leqslant\Sigma_i W(\boldsymbol{A}_i)+N=W(\boldsymbol{A})+N$.



(ii) If a non-splitting $\boldsymbol{A}$ evolves into $\boldsymbol{a}$ of N splittings, then $W(\boldsymbol{a})\leqslant W(\boldsymbol{A})-N$.

Proof. FRT implies the raw widths of $\boldsymbol{a}$ and $\boldsymbol{A}$ are equal, i.e. $\mathbb{W}(\boldsymbol{a})=\mathbb{W}(\boldsymbol{A})$. Hence $W(\boldsymbol{a})+N=\Sigma_{i=1}^{N+1}W(\boldsymbol{b}_i)+N\leqslant\mathbb{W}(\boldsymbol{a})$, where $\boldsymbol{b}_i$ are the split subpatterns of $\boldsymbol{a}$.

(iii) If a particle $\boldsymbol{A}=\Sigma_{i=1}^{M+1}\boldsymbol{B}_i$ ($\boldsymbol{B}_i$ are non-splitting particles) has exactly M splittings separating $\boldsymbol{B}_i$ and $\boldsymbol{A}$ evolves into a single non-splitting particle $\boldsymbol{a}$ in the next step, then $W(\boldsymbol{a})\leqslant W(\boldsymbol{A})+M$.

Proof. For $M=1$, without loss of generality, we assume that $\boldsymbol{B}_i$ and $\boldsymbol{b}_i$ are not zero. We can see from the FRT that there is a *bit* position α in $\boldsymbol{B}_2$ (see the diagram below, a vertical arrow indicates the position mapping for the evolution) such that a subpattern $\boldsymbol{C}_2$ starting from that position to the end of $\boldsymbol{B}_2$ evolves into $\boldsymbol{c}_2$ with, due to (i), $W(\boldsymbol{c}_2) \leqslant W(\boldsymbol{C}_2) \leqslant W(\boldsymbol{B}_2)$. Therefore $W(\boldsymbol{b}_1)+W(\boldsymbol{c}_2) \leqslant W(\boldsymbol{B}_1)+W(\boldsymbol{C}_2) \leqslant W(\boldsymbol{B}_1)+W(\boldsymbol{B}_2)$ and there are exactly $(r+1)$ bits separating $\boldsymbol{b}_1$ and $\boldsymbol{c}_2$, and hence (iii) is proved in this simplest case. For a general $M>1$, we prove it inductively. If we use the FRT again to the subpattern $\boldsymbol{A}'$ starting from the position α to the end of $\boldsymbol{A}$, i.e. $\boldsymbol{C}_2+\boldsymbol{B}_3+,\ldots,+\boldsymbol{B}_{M+1}$, then the above process can go on to absorb a further particle on the RHS of $\boldsymbol{B}_2$. If by starting at position α, $\boldsymbol{C}_2$, $\boldsymbol{B}_3,\ldots,\boldsymbol{B}_m$ $(m \leqslant M+1)$ together form a new non-splitting particle $\boldsymbol{D}$ split from the left-over particles on the RHS, then from (i) $W(\boldsymbol{D}) \leqslant W(\boldsymbol{B}_2)+\ldots+W(\boldsymbol{B}_m)+m-2$, and we then apply the above procedure to $\boldsymbol{D}+\boldsymbol{B}_{m+1}+\boldsymbol{B}_{M+1}$. In either case, the absorption of *one* non-splitting particle $\boldsymbol{B}_i$ adds 1 to the total width of the new pattern state. This proves (iii). We note that this procedure remains valid even if $\boldsymbol{b}_1$, $\boldsymbol{c}_2$ etc. in $\boldsymbol{a}$ has splittings.

$$
\begin{array}{ll}
 & \qquad\qquad\qquad\qquad \text{splitting} \qquad\quad \alpha \\
\text{A:} & \boxed{1} \leftarrow ?\ \text{—}B_1 \rightarrow \boxed{0}\,\text{o..o...o}\,\boxed{1} \leftarrow ?\ \bullet\quad B_2 \rightarrow \boxed{0}\,\text{o.....o}\,\boxed{1} \leftarrow B_{N+1} \rightarrow \boxed{0} \\
 & \quad\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow\!|\!\leftarrow C_2 \rightarrow| \\
 & \qquad\qquad\qquad\qquad\qquad |\!\leftarrow r+1 \rightarrow\!| \\
\text{a:} & \quad \boxed{1} \leftarrow b_1 \longrightarrow \boxed{0}\,\text{o...o....?}\,\boxed{*} \leftarrow c_2 \longrightarrow \boxed{0}\,*\ldots\ldots\ldots\ldots*\,\boxed{0}
\end{array}
$$

(iv) In general, a pattern state $\boldsymbol{A}$ consists of a combination of L subpatterns $\boldsymbol{A}_i$ where $\boldsymbol{A}_i$ are combinations of particles which will all link up at their current splittings inside $\boldsymbol{A}_i$, totalling M_i linking-ups, and furthermore create N_i new splittings. These $\boldsymbol{A}_i$ can also be assumed to be split and remain so from each other in the next step of evolution. By applying (iii) (or rather the procedure there) to $\boldsymbol{A}_i$ because we can treat $\boldsymbol{A}_i$ separately, we have $W(\boldsymbol{A}) \leqslant \Sigma_{i=1}^{L} W(\boldsymbol{A}_i)+\Sigma_{i=1}^{L}(M_i-N_i)$. This proves case (b) and also completes the proof of theorem 2. □

We now define a compound pattern by adjacently combining all the non-splitting subpatterns of a particle $\boldsymbol{A}$, i.e. their first boxed bit (inclusive) through the last one (exclusive), of $\boldsymbol{a}^t$. In other words, a pattern state $\boldsymbol{A}$ of the form

$$
\begin{array}{l}
|\!\leftarrow a_1 \longrightarrow\!|\ \text{splits here}\ |\!\leftarrow a_2 \longrightarrow\!| \qquad\qquad |\!\leftarrow a_m \longrightarrow\!| \\
\boxed{1}\,\alpha_1\cdots\cdots\beta_1\,\boxed{0}\,\text{o.........o}\,\boxed{1}\,\alpha_2\cdots\cdots\beta_2\,\boxed{0}\,\text{o.....o}\,\boxed{1}\,\alpha_m\cdots\cdots\beta_m\,\boxed{0}
\end{array}
$$

will define a compound pattern state $\mathscr{C}(A)$ by

$$
\boxed{1}\,\alpha_1\cdots\cdots\beta_1\,\boxed{1}\,\alpha_2\cdots\cdots\beta_2\,\boxed{1}\,\alpha_3\cdots\cdots\beta_{m-1}\,\boxed{1}\,\alpha_m\cdots\cdots\beta_m
$$

From theorem $2(a)$ we know that the widths of the compound patterns are bounded, thus some compound pattern will reappear after sufficiently many steps. Hence for any initial pattern state $\boldsymbol{a}^0$, as time $t \to \infty$, there is a compound state $\mathscr{C}(\boldsymbol{a}^T)$ at $t = T$ which will appear infinitely many times for $t > T$. This lays the foundation for the following:

Theorem 3 (asymptotic theorem). For any seed $\boldsymbol{A}$, there is a finite collection of particle patterns $\boldsymbol{A}_i$ such that as $t \to \infty$ they will reappear as the exact collection, i.e. $\Sigma_i \boldsymbol{A}_i$, of split subpatterns for infinitely many times.

The above theorem is a typical scenario of a seed evolving into a collection of periodic particles with faster ones on the right. We now prove theorem 3. Let $\boldsymbol{C} = \boldsymbol{B}_1, \ldots, \boldsymbol{B}_m$ be the infinitely recurring compound pattern, induced by $\boldsymbol{A}$ as we discussed before, occurring at $t = t_i$ for $i\ (\in \mathbb{Z}) \geqslant 0$, where $\boldsymbol{B}_j$ are BSs. We call $\mathscr{P}(\boldsymbol{C}) = (\boldsymbol{B}_1, \ldots, \boldsymbol{B}_{j_1})$ $(\boldsymbol{B}_{j_1+1}, \ldots, \boldsymbol{B}_{j_2})$ $(\ldots)$ $(\boldsymbol{B}_{j_k+1}, \ldots, \boldsymbol{B}_m)$ a particle grouping of $\boldsymbol{C}$ into a set of particles, if at $t = t_i$ all the BSs inside the same pair of parentheses form one single non-splitting particle and different particles thus formed are split from each other. Since there are again only a finite number of different such groupings in the same compound pattern $\boldsymbol{C}$, then one particular grouping will reoccur infinitely many times. This proves theorem 3.

Let $\boldsymbol{D}_i = \boldsymbol{B}_{j_i+1}, \ldots, \boldsymbol{B}_{j_{i+1}}$ $(i = 1, \ldots, n)$ be the corresponding particles of the above grouping, we add a new grouping of $\boldsymbol{D}_i$s by their distances. As t goes to infinity, if the distance between $\boldsymbol{D}_i$ and $\boldsymbol{D}_{i+1}$ is not bounded, we insert a pair of group separators ')(' between them. By adding a pair of parentheses on both sides of $\mathscr{P}(\boldsymbol{C})$, we obtain *a distance grouping of* $\boldsymbol{D}_1, \ldots, \boldsymbol{D}_n$ *into* $\boldsymbol{A}_i = \boldsymbol{D}_{q_i+1} \ldots \boldsymbol{D}_{q_{i+1}}$. *Hence if we allow* $\boldsymbol{A}_i$ *in* theorem 3 to be relaxed to a summation of a finite number of particles, then the theorem remains valid with the distances separating the $\boldsymbol{A}_i$ also tending to infinity.

Notice that if bounded subparticles $\boldsymbol{A}_i$ of $\boldsymbol{A}$ are well split from each other with distances between them tending to infinity, then theorem $2(c, d)$ applies to each $\boldsymbol{A}_i$ separately and hence $\boldsymbol{A}$ evolves into a collection of periodic subparticles with faster ones on the right. In fact, our computer simulation manifested this by showing all sampled seeds having no restrictions disintegrating into periodic particles lined up in the order of their speeds. This kind of phenomenon is well *observed* for PSTs CA (3) [1, 7].

Similar to the case of CA (4), we associate $\boldsymbol{a}^t$ with an energy [7, 11]

$$E(\boldsymbol{a}^t) = \sum_{i=0}^{r} \left(\sum_{j=-\infty}^{\infty} |a^t_{i+r+1+j} - a^t_{i+j}| \right) \equiv \sum_{i=-\infty}^{\infty} |a^t_i - a^t_{i-r-1}|. \tag{10}$$

Obviously if $\boldsymbol{a}^t$ and $\boldsymbol{a}^{t+1}$ both have no splittings, then $E(\boldsymbol{a}^{t+1}) = E(\boldsymbol{a}^t)$ (cf [7, 11]) because the bits changed during a one-step evolution are simultaneously conjugate in a distance of $(r+1)$ which will not alter the expression (10). Also if there are $r+1$ consecutive zeros or more between particles $\boldsymbol{b}$ and $\boldsymbol{c}$, then $E(\boldsymbol{b}+\boldsymbol{c}) = E(\boldsymbol{b}) + E(\boldsymbol{c})$. Thus if $\boldsymbol{a}$ splits into $\Sigma_b \boldsymbol{b}_k$, then $E(\boldsymbol{a}) = \Sigma_k E(\boldsymbol{b}_k)$. Notice that *immediately* before and after a linking-up, there are always at least $r+1$ consecutive zeros in between, hence energy is preserved during particle evolution. Therefore we have now proved:

Theorem 4 (energy theorem). The energy given by (10) is conserved for CA (5).

We note that a BS $= (\mathbf{1}0 \ldots 0)$ is a 'zero energy header' which can be put on the very front of a non-zero particle without changing its energy. This is also the reason for non-invertibility of the particle evolution.

4. Particle collision

For particle collisions, we (as in [1]) only consider *proper* interactions, i.e. the particles are placed sufficiently far apart before interaction. In [1] a so-called determinant is used for PSTs CA to calculate the upper bound of the number of proper interactions. Here, however, we shall give an algorithm for all different proper interactions, which makes it tractable to simulate collisions exhaustively within any given range, instead of choosing random seeds as in [1]. Let x be a periodic particle and $\mathcal{O}(x)$ be the state orbit of the evolution of x, we choose *lexically* the largest state X of $\mathcal{O}(x)$ as the canonical form which represents the particle.

Theorem 5 (collision theorem). Suppose X and Y are two canonical particles (X on the LHS of Y), let p_1 and p_2, d_1 and d_2, v_1 and v_2 be the corresponding periods, displacements and speeds respectively with $v_1 = d_1/p_1 > v_2 = d_2/p_2$, and let $q = (p_1, p_2)$ be the greatest common factor of p_1 and p_2, i.e. $p_i = qq_i$ and q_1 and q_2 are prime to each other, then there are exactly $\Delta = p_2 d_1 - p_1 d_2$ *different* proper interactions between the two particles, explicitly given by

$$\sigma^j(X) \xleftrightarrow{m+l_0} Y \qquad 0 \leq j < q \qquad 0 \leq m < M \equiv q_2 d_1 - q_2 d_2 \tag{11}$$

where $\sigma(x)$ means the next state of x in $\mathcal{O}(x)$, $m + l_0$ represents the distance (i.e. bit count) of the last box of $\sigma^j(x)$ to the first box of Y and $l_0 > d_2 + r + 1$ is any fixed integer.

We note that two interactions are considered the same if, before reaching collision, one pair of particles will evolve into the exact state of the other up to a global spatial displacement. Also the definition of canonical form is not important and (11) is indeed valid for any states $X \in \mathcal{O}(x)$ and $Y \in \mathcal{O}(y)$. Moreover the above results remain the same whether or not the displacement and the speed are with respect to a moving frame as long as they are in the same frame. In fact, if we are on another moving frame with a constant speed c, then the new speeds are $v_i' = d_i/p_i + c \equiv d_i'/p_i$ and it is obvious Δ and M in (11) are invariant with the new displacement d_i'. We now prove theorem 5. Notice that any pair of periodic particles x and y posed for a proper collision can always evolve into or from

$$x' \xleftrightarrow{m+l_0} Y \qquad x' \in \mathcal{O}(x) \qquad 0 \leq m < M \tag{12}$$

because after qq_1q_2 steps (thus travelling M positions) from (12) pattern x' and Y will reappear. Hence all interactions come from (12), totalling $p_1 M$ collisions. However as x' approaches Y via evolution within (12) from $m = M - 1$, Y will appear exactly q_1 times via pairing $\sigma^{ip_2}(x') \leftrightarrow Y, 0 \leq i < q_1$. This therefore implies there are $\Delta\ (= p_1 M/q_1)$ proper interactions. We note that our pairing (11) also applies to PSTs CA (3) where Δ is precisely the determinant proposed in [1].

We now show that the Δ collisions paired by (11) are all different and thus exhaust all the proper interactions between particles X and Y. First of all, we notice that in pairing $x \xleftrightarrow{m+l_0} Y, 0 \leq m < M$, there are no duplications. Because otherwise from the above there would be an i such that $|i| < q_1$ and $x = \sigma^{ip_2}(x)$, i.e. $ip_2 \equiv 0 \bmod(p_1)$ which contradicts $(q_1, q_2) = 1$. No duplications can come from (11) for $0 < j < q$ either. In fact, if $\sigma^k(X) \xleftrightarrow{m'+l_0} Y$ $(0 \leq k < q)$ and $\sigma^j(X) \xleftrightarrow{m+l_0} Y$ $(0 < j < q)$ were essentially the same, i.e. one evolves into the other, then we would have $\sigma^j(X)=(\sigma^k(X))^{ip_2}=\sigma^{kip_2}(X)$. Thus $j=\alpha p_1+\beta p_2<q$ for some α and $\beta\in\mathbb{Z}$ which contradicts the fact that the greatest common factor $q=(p_1,p_2)$ is the smallest positive integers satisfying this relation for all α and β. Hence the proof is completed.

We recall that all non-zero 1-BS strings, in contrast to PSTs model, are non-trivial simple particles under (5). Nevertheless their interactions are still solitonic and can be proved rigorously by closely following the proof of Fokas *et al* [6] for CA (4) and by keeping in mind that our splitting condition coincides with that in [6] when only 1-BS particles are involved. In fact a very interesting analytical result can be imported directly from that of [6, 7] with only minor modifications.

Suppose a periodic particle does not split at any time and is given at $t=0$ by $0A^1A^2\ldots A^L0$ with $l_0, l_1,\ldots, l_L$, the number of 1s in the BSs: A^1, $A^1\oplus A^2$, $A^2\oplus A^3,\ldots, A^{L-1}\oplus A^L$, A^L, where $\oplus$ denotes [7] the *exclusive or* operation, then the particle pattern will reappear in $P=l_0+\ldots+l_L$ steps of evolution. Furthermore, the displacement at $t=i$ on the moving frame (to the left with speed $=r+1$) is given by $D_i=d_{i+1}-d_1$ where

$$d_i=\begin{cases} k(r+1)+d_j^k & i=l_0+\ldots+l_{k-1}+j \quad 1\leqslant j\leqslant l_k \quad 0\leqslant k\leqslant L \quad l_{-1}=0\\ mL(r+1)+d_j & i=m(l_0+\ldots+l_L)+j \quad 1\leqslant j\leqslant l_0+\ldots+l_L \quad m=1,2,\ldots\end{cases}$$

and $d_j^k=j$th 1 of the BS $A^k\oplus A^{k+1}$ $(k=0,\ldots,L)$, with $A^0=A^{L+1}\equiv 0$.

5. Model comparison and simulation statistics

One of the important features of our CA (5) is that *any* seed of solitonic interaction under PSTs CA (3) will also be a seed of solitonic interaction under (5) and the evolutions from the seed differ only by a constant moving frame. However, the opposite is not true as we shall see from figures 1-4. Hence CA (5) contains richer solitonic phenomena. In fact, suppose we place boxes to a seed by our FRT and that for (4) [4], if they coincide then no energy will be lost during the *next step* of evolution. Otherwise there will be a BS $=10\ldots0$ at the end of at least one particle, which means it will lose energy [7] under (4). Obviously any solitonic collision under PSTs CA will not lose energy and therefore will also be preserved under our new CA (5) up to a moving frame. Other features such as breathers and the so-called premature splittings for (4) are well observed for (5) too [16]. Unlike other models [3], however, turbulence does not occur for CA (5).

We now establish some simulation statistics and their comparison under (4) and (5). We note that two particles of the same speed are excluded from our proper interactions. But first of all, let us show some of the solitonic interactions under (5) that would deviate under (4). All figures depict evolution on a left moving frame. Figure 1 shows a 1-BS particle colliding with a 2-BS and a 3-BS particle at the same time and re-emerging after interaction. Notice that no splitting occurs at $t=8$ thus all three particles are in *full* collision. Also two of the three particles would evolve into zero before collision under CA (4). In figure 2, two simple particles are posed to fire at a 3-BS particle at the same time and again reappear later on. In this case, all three particles are also non-trivial periodic particle seeds under (4) but will lose energy *during* the collision and thus be unable to recover their original patterns. In our third example, we present an occasional feature shared by both (4) and, less often, (5),

Figure 1. Solitonic collision for equation (5) with left-moving frame speed $= r = 3$. Two particle would nullify under equation (4).

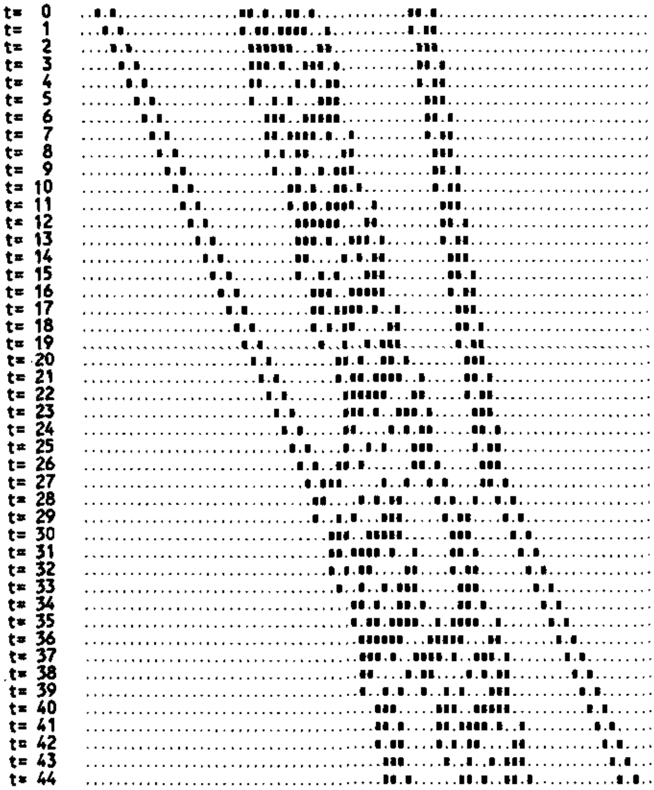


Figure 2. Solitonic collision for equation (5) with left-moving frame speed $= r = 3$. Collision would be non-solitonic under equation (4).

which so far seems to have escaped general observation: during a collision of two particles, one particle may reappear while the other changes into new form with equal or less energy. It is largely due to the exhaustiveness of theorem 5 that such rare cases (and phaseshift characteristics to be explained later on) do not slip away unnoticed. Figure 3 shows such a feature for (4) which under (5) will still be solitonic with 11001100100000000111 at $t = 0$ evolving into pattern 11100000000000110011001 at $t = 46$. Finally in table 1, we give the distribution of all the different types of interactions for CA (5), as well as for CA (4) for comparison. The simulation program is written in C for efficiency, but an exhaustive checking for higher range would still come up against

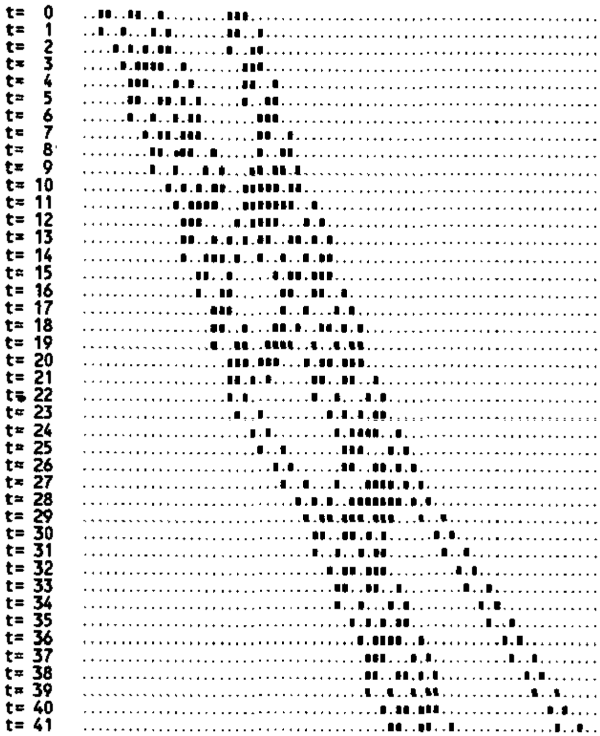


Figure 3. Energy loss for equation (4) with $r=4$ and left-moving frame speed $=3$. One particle reappears after interaction. Collision would be solitonic under equation (5).

the computer (SUN workstation) speed limitation. We note that table 1 shows a higher percentage of solitonic collisions in CA (5) than in CA (4). No collisions of 1-BS versus 1-BS are reported there because they are all solitonic. Also no proper interactions exist for 2-BS versus 2-BS under CA (4) with $r=2$ because all those 2-BS particles have the same speed.

It is noticed [1] via simulations that in a solitonic collision under CA (3) the fast (rightwards) particle cannot be shifted to the left, and the slow particle cannot be shifted to the right. This kind of observation is typical of computerization taking initiatives when theory fails to proceed. Our more extensive simulations show, however, that the above property, while true for CA (3), may fail in some rare cases (less than 0.04% of total collisions sampled in table 1) for our new CA (5). In other words, the fast particle may be dragged backward and the slow one pushed forward. Figure 4 depicts such an example where two collisions of the same pair of particles with different initial distances result in different signs of phaseshifts. In general, the phaseshifts of a solitonic (two particle) collision depend on the exact pattern configuration of each particle in the initial state and the distance between them at the time. Nevertheless we do have a new interesting empirical property: if two particles reappear travelling apart after their interaction, the exact pattern of each particle in the seed before collision will also *simultaneously* reappear later on (see figures 1 and 2). That is, the exact patterns of each particle before interaction ($t=0$) can and will reappear at the same time.

Table 1. Collision statistics.

		1-BS versus	2-BS	2-BS versus	2-BS
Rad		CA (5)	CA (4)	CA (5)	CA (4)
2	M	37	1	29	0
	O	0	0	0	0
	N	0	0	3	0
	C	1	2	4	0
	T	38	3	36	0
	%	97	33	81	
3	M	458	110	1172	168
	O	4	1	5	2
	N	10	24	266	78
	C	4	14	15	94
	T	476	149	1458	342
	%	96	74	80	49
4	M	4512	2045	30310	10499
	O	22	28	35	30
	N	120	295	4549	4525
	C	12	52	1034	1572
	T	4666	2420	35928	16626
	%	97	85	84	63

Rad = radius, M = (mirror) reverse match, O = one particle matched only, C = seed collapses into a periodic particle, N = no particles in seed reappear after collision, T = total collision number, % = percentage of solitonic interactions against the total ones.

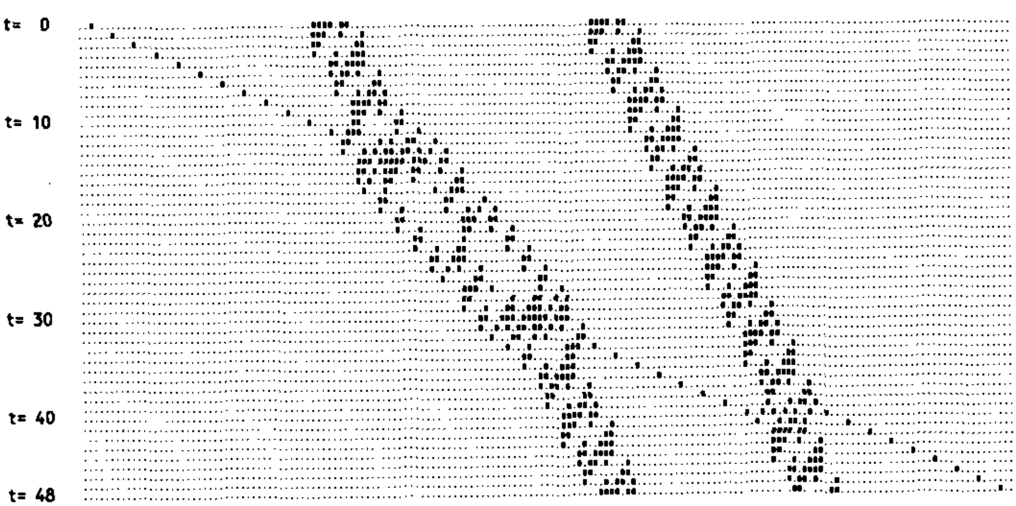


Figure 4. Different phaseshift directions for equation (5) with left-moving frame speed = r = 4 for the same pair of particles depending on the initial separating distances.

References

[1] Park J K, Steiglitz K and Thurston W P 1986 *Physica* **19D** 423

[2] Farmer D, Toffoli T and Wolfram S 1984 *Cellular Automata* (Amsterdam: North-Holland)

[3] Aizawa Y, Nishikawa I and Kaneko K 1990 *Physica* **45D** 307

[4] Papatheodorou T S, Ablowitz M J and Saridakis Y G 1988 *Stud. Appl. Math.* **79** 173

[5] Papatheodorou T S and Fokas A S 1989 *Stud. Appl. Math.* **80** 165

[6] Fokas A S, Papadopoulou E P, Saridakis Y G and Ablowitz M J 1989 *Stud. Appl. Math.* **81** 153

[7] Fokas A S, Papadopoulou E P and Saridakis Y G 1990 *Physica* **41D** 297

[8] Fokas A S, Papadopoulou E and Saridakis Y 1990 *Phys. Lett.* **147A** 369

[9] Takahashi D and Satsuma J 1990 *J. Phys. Soc. Japan* **59** 3514

[10] Ablowitz M, Keisera J and Takhtajan L 1991 *Phys. Rev.* A **44** 6909

[11] Golberg C H 1988 *Complex Systems* **2** 91

[12] Fordy A P (ed) 1990 *Soliton Theory: A Survey of Results* (Manchester: Manchester University Press)

[13] Jiang Z and Rauch-Wojciechowski S 1991 *J. Math. Phys.* **32** 1720

[14] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)

[15] Jiang Z 1989 *Inv. Prob.* **5** 349; 1990 *Phys. Lett.* **148A** 57

[16] Jiang Z and Bullough R K 1992 *Proc. NEEDS-91* (Singapore: World Scientific)